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Analytic structure and chaotic dynamics of the damped driven Toda oscillator

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The singularity structure exhibited by the solution of the damped driven Toda oscillator in the complex time $(t-)$ plane is investigated through Painlevé $(P-)$ analysis. We find that there exists a specific parametric choice for which the free but *damped* Toda oscillator possesses the $P-$ property and hence is likely to be integrable. We present the exact solution corresponding to this integrable choice. In the nonintegrable regime, we show that the singularities exhibit locally a complicated, clustered, two-armed infinite-sheeted Riemann structure in the complex $t-$ plane. Further, we have analyzed numerically the global singularity structure in the complex $t-$ plane (i.e., analytic structure) corresponding to the real time chaotic dynamics exhibited by the system. From the investigations, we observe that the global singularity structure exhibits a ''chimneylike'' pattern in which the width at the bottom of the chimney decreases and the singularities tend to cluster at the top of the chimney, in the complex $t-$ plane corresponding to the real-time chaotic dynamics exhibited by the system, as the control parameter is varied. [S1063-651X(97)10703-6]

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We consider the general form of the equation of motion of the damped driven Toda oscillator [1], given by

$$\ddot{x}+d\dot{x}+\alpha e^{x}-\beta=f\cos\omega t, \tag{1}$$

where d is the viscous damping parameter and f and ω are, respectively, the amplitude and frequency of the external periodic force. The parameters α and β are associated with the potential. Since the potential is asymmetric, the system described by Eq. (1) can serve as a typical model for asymmetric oscillators. It has already been used to describe a nonlinear electronic circuit [2]. Here we wish to investigate both the integrability and nonintegrability aspects of the system described by Eq. (1) through Painlevé $(P-)$ analysis [3–13] by examining the nature of the singularities exhibited by the solution in the complex time plane. The dynamical systems having nonpolynomial terms in their equations of motion such as that in Eq. (1) can be conveniently investigated for the singularity structure by converting them into polynomial or rational forms. This is done by making an exponential transformation of the dependent variables. However, we note that very few dynamical systems that are of nonpolynomial type—such as the Arnold-Beltrami-Childress (ABC) flows [10], the sine-Gordon equation, the driven pendulum [11], and the damped driven Morse oscillator [12]—have been studied in this way. In this paper, after making such an exponential transformation we show that there exists a specific choice of the parameter $\beta=-2d^2$ for which the free $(f=0)$ but *damped* Toda oscillator is free from movable critical points and hence is likely to be integrable. We present the exact solution corresponding to the above parametric choice. In the nonintegrable regime, we show that the singularities exhibit locally a complicated, clustered, two-armed, infinite-sheeted Riemann structure in the complex $t-$ plane. Further, we analyze numerically the global singularity structure in the complex $t-$ plane corresponding to the real time chaotic dynamics exhibited by the system.

Introducing the transformation

$$y=e^{x}, \tag{2}$$

Eq. (1) reduces to

$$y\ddot{y}-\dot{y}^2+dy\dot{y}+\alpha y^3-\beta y^2-fy^2\cos\omega t=0. \tag{3}$$

We will analyze the singularity structure of the solution to this equation. The general solution to Eq. (3) can be represented locally as a Laurent series of the form

$$y=\sum_{j=0}^{\infty} a_j\tau^{j-2}, \quad \tau=(t-t_0)\to 0 \tag{4}$$

about an arbitrary movable singularity t_0, in which one of the a_j's must be arbitrary in addition to t_0. Substituting the anzatz (4) into Eq. (3), we obtain the recursion relations for the a_j:

$$\sum_r \Big[(j-r-2)(j-2r-1)a_{j-r}a_r+d(j-r-3)a_{j-r-1}a_r + \alpha\sum_p a_{j-r}a_{r-p}a_p-\beta a_{j-r-2}a_r - f\sum_p G_{j-r-2}a_{r-p}a_p\Big]=0, \quad 0\leqslant p\leqslant r\leqslant j, \tag{5}$$

where $G(t)=\cos\omega t$ and $G_n=(1/n!)[\partial^n G(t)]/(\partial t^n)|_{t=t_0}$. From Eq. (5) one obtains

$$j=0: a_0=-2/\alpha, \tag{6a}$$

$$j=1{:}a_1=2d/\alpha, \tag{6b}$$

$$j=2{:}0.a_2-(2d^2+\beta+f\cos\omega t_0)a_0=0. \tag{6c}$$

Thus, from Eq. (6c) it is evident that a_2 is the needed arbitrary coefficient. But for a_2 to be arbitrary, an inconsistency arises in Eq. (6c). This means that, in general, for $f\neq 0$ the Laurent series (4) has to be modified. However, when $f=0$, two possibilities arise from Eq. (6c) for a_2 to be arbitrary, as follows.

(a) $d=0$ and $\beta=0$. This condition implies that the equation of motion of the free undamped oscillator may be converted to a first-order Bernoulli equation and trivially integrated.

(b) $2d^2+\beta=0$. Here the equation of the free damped oscillator $[f=0$ in (1)] possesses the $P-$ property for the specific value of the parameter β given by

$$\beta=-2d^2. \tag{7}$$

For the general case $d\neq 0$, $\beta\neq 0$, and $f\neq 0$, the arbitrariness of a_2 can be recaptured by modifying the anzatz (4) and introducing logarithmic terms in Eq. (4) through the psi series [7]

$$y=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}\tau^{j-2}(\tau^2\ln\tau)^k. \tag{8}$$

Then the recursion relations for the a_{jk} for Eq. (3) become

$$\begin{aligned}\sum_{r,s}\Big\{&(j-r+2k-2s-2)(j-2r+2k-4s-1)a_{j-r,k-s}a_{rs}\\&+[(2j-2r+4k-4s-5)(k-2s+1)-s]\\&\times a_{j-r-2,k-s+1}a_{rs}+(k-s+2)(k-2s+1)\\&\times a_{j-r-4,k-s+2}a_{rs}+d(j-r+2k-2s-3)\\&\times a_{j-r-1,k-s}a_{rs}+d(k-s+1)a_{j-r-3,k-s+1}a_{rs}\\&+\alpha\sum_{p,q} a_{j-r,k-s}a_{r-p,s-q}a_{pq}-\beta a_{j-r-2,k-s}a_{rs}-f\\&\times\sum_{p} G_{j-r-2}a_{r-p,k-s}a_{ps}\Big\}=0,\\&0\leqslant p\leqslant r\leqslant j,0\leqslant q\leqslant s\leqslant k.\end{aligned} \tag{9}$$

The values of the coefficients a_{00} and a_{10} are given by $a_{00}=-2/\alpha$ and $a_{10}=2d/\alpha$. For a_{20} to be arbitrary we now have

$$0.a_{20}-a_{01}-(2d^2+\beta+f\cos\omega t_0)a_{00}=0, \tag{10}$$

which means that

$$a_{01}=2(2d^2+\beta+f\cos\omega t_0)/\alpha. \tag{11}$$

From Eq. (8) we see that the singularity t_0 is no longer a movable pole but is, instead, a movable logarithmic branch point and Eq. (3) is not of $P-$ type. Thus the system (1) is, in general, nonintegrable except when (i) $d=0,\beta=0,f=0$ and (ii) $\beta=-2d^2,f=0$.

Now we proceed to obtain the exact solutions of the free damped Toda oscillator with $\beta=-2d^2$. Then the equation of motion (3) becomes

$$y\ddot{y}-\dot{y}^2+dy\dot{y}+\alpha y^3+2d^2y^2=0. \tag{12}$$

By substituting $\dot{y}=yu$ and its derivative in Eq. (12), we get

$$\ddot{u}-(u-d)\dot{u}-2d^2u-du^2=0. \tag{13}$$

Introducing new variables $u=-R-d$ and $z=t/2$, Eq. (13) can be transformed into

$$R''=-2RR'-4dR'-4dR^2+4d^3, \tag{14}$$

where the prime indicates differentiation with respect to z. The first integral of Eq. (14) is given by [14]

$$R'+R^2=v, \tag{15a}$$

where

$$v'=-4dv+4d^3. \tag{15b}$$

Equation (15b) can be readily solved for v and inserted into (15a). When this is followed by the substitution $R=S'/S$, it results in the following linear equation:

$$S''-[d^2+c_0\exp(-4dz)]S=0, \tag{16}$$

where c_0 is an integration constant. Equation (16) may be put into a more well known form by changing the dependent variable $S(z)$ as $S(z)=T(r)$, where $r=\exp(-2dz)$, to give

$$r^2T''+rT'-\left[\frac{1}{4}+\frac{c_0}{4d^2}r^2\right]T=0. \tag{17}$$

Equation (17) is a standard Bessel function equation [15], one of whose solutions is $T=Z_{1/2}(r\mu)$, where $\mu=\sqrt{-c_0/4d^2}$ and $Z_{1/2}$ is a Bessel function of one-half order. Now the solution y can be written as

$$y=y_0\exp(-dt)/[Z_{1/2}(\mu\exp(-dt)]^2. \tag{18}$$

In Eq. (18), y_0 is another integration constant. Equation (18) represents an exponentiallike decaying solution of (12), as expected.

In order to study the analytic structure of the solution of Eq. (3), we now look for a closed set of recursion relations among the a_{jk}'s given by Eq. (9). These turn out to be the set $a_{0k}, k=0,1,2,\ldots$, which satisfy

$$\begin{aligned}\sum_{s}\Big\{&[4(k-s)(k-s-1)-4s(k-s)+4s-2(k-s)\\&+2]a_{0,k-s}a_{0s}+\alpha\sum_{q} a_{0,k-s}a_{0,s-q}a_{0q}\Big\}=0.\end{aligned} \tag{19}$$

Introducing now the generating function

$$\Theta(z)=\sum_{k=0}^{\infty} a_{0k}z^k, \tag{20}$$

where the z is a function of τ, the following differential equation for $\Theta(z)$ is obtained:

$$4z^2\Theta\Theta'' - 4z^2\Theta'^2 + 2z\Theta\Theta' + 2\Theta^2 + \alpha\Theta^3 = 0, \quad (21)$$

where the prime denotes differentiation with respect to z. Since in the limit $\tau \to 0$, the most dominant terms in the psi series (8) involve powers of $\tau^2 \ln\tau$ only, we can obtain (21) in a more direct way by substituting

$$y(t) = \frac{1}{\tau^2}\Theta(z), \quad (22)$$

where

$$z = \tau^2 \ln\tau, \quad (23)$$

into (3). Thus (21) can be regarded as the original equation (3) rescaled in the neighborhood of a given singularity t_0. Further, it is a straightforward exercise to show that Eq. (21) possesses the Painlevé property with $\Theta(z)$ having the local expansion

$$\Theta(z) = \sum_{j=0}^{\infty} A_j (z - z_0)^{j-2} \quad (24)$$

in which A_2 and z_0 are the arbitrary parameters.

We can also see that Eq. (21) can be integrated exactly by making the substitution

$$\Theta(z) = \xi^2 f(\xi), \quad \xi = \sqrt{z} \quad (25)$$

in Eq. (21) so that we get

$$ff'' - f'^2 + \alpha f^3 = 0, \quad (26)$$

where the prime refers to differentiation with respect to ξ. The first integral of (26) is given by [14]

$$f'^2 = -2\alpha f^3 + I_1 f^2, \quad (27)$$

where the value of the integration constant I_1 can be determined as $I_1 = 6\alpha a_{01} = 12(2d^2 + \beta + f\cos\omega t_0)$. By a simple transformation

$$f(\xi) = [1 - g^2(\xi)] I_1 / 2\alpha, \quad (28)$$

Eq. (27) is reduced to a simple first-order nonlinear ordinary differential equation

$$g' = \tfrac{1}{2}\sqrt{I_1}[g^2 - 1]. \quad (29)$$

Equation (28) can be readily integrated and its solution is given by

$$g(\xi) = -\tanh\left[\frac{1}{2}\sqrt{I_1}(\xi - \xi_0)\right], \quad (30)$$

where ξ_0 is the arbitrary integration constant. Choosing $\xi_0 = 0$ for convenience, we can write

$$f(\xi) = \frac{I_1}{2\alpha}\operatorname{sech}^2\left[\frac{1}{2}\sqrt{I_1}\xi\right]. \quad (31)$$

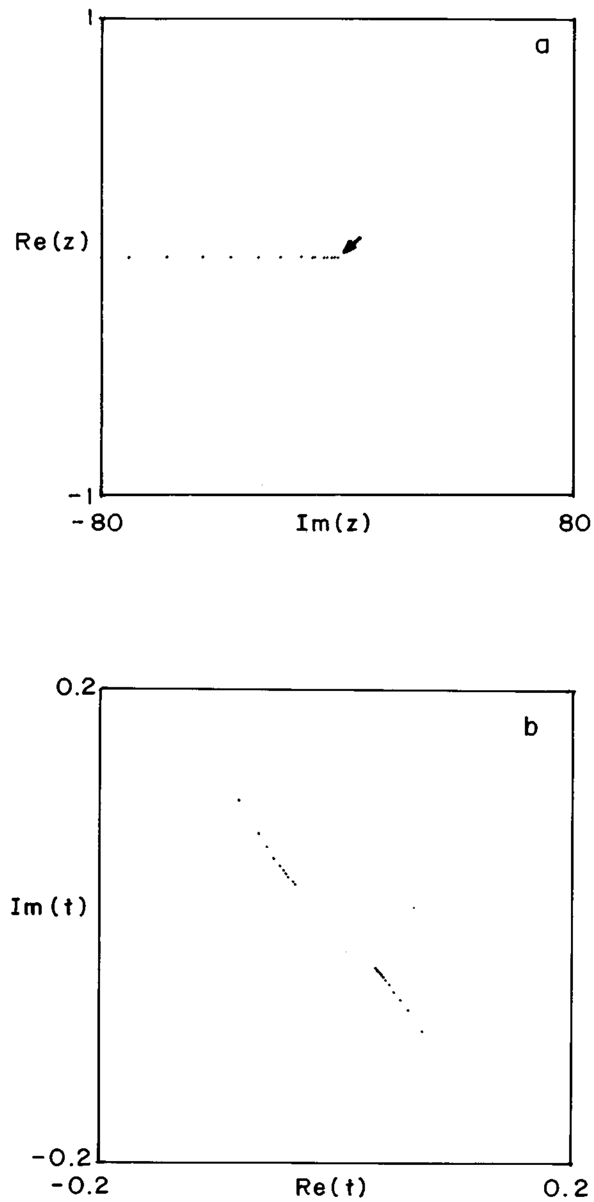

FIG. 1. Local singularity structure in the complex (a) $z-$ plane ($z = t^2 \ln t$) given by Eq. (33) and (b) $t-$ plane in the neighborhood of the marked singularity in (a) determined from the analytic mapping (38) and (39) for $d = 0.25$, $\alpha = 1.0$, $\beta = 1.0$, $\omega = 2.0$, and $f = 0.5$. The value of $m = 0, \pm 1, \pm 2, \ldots, \pm 9, \pm 10$ for (a) and $m = 0$ for (b).

It is evident that $f(\xi)$ has poles of second order which are situated at the discrete points

$$\xi_m = i\frac{\pi}{\sqrt{I_1}}(2m+1), \quad m \in Z \quad (32)$$

in the complex ξ plane, where m denotes the lattice site integer.

The singularity positions in the $z-$ plane can be obtained from [c.f. Eq. (25)] the pole positions of ξ_m as

$$z_m = -\pi(2m+1)^2 / 12(2d^2 + \beta + f\cos\omega t_0). \quad (33)$$

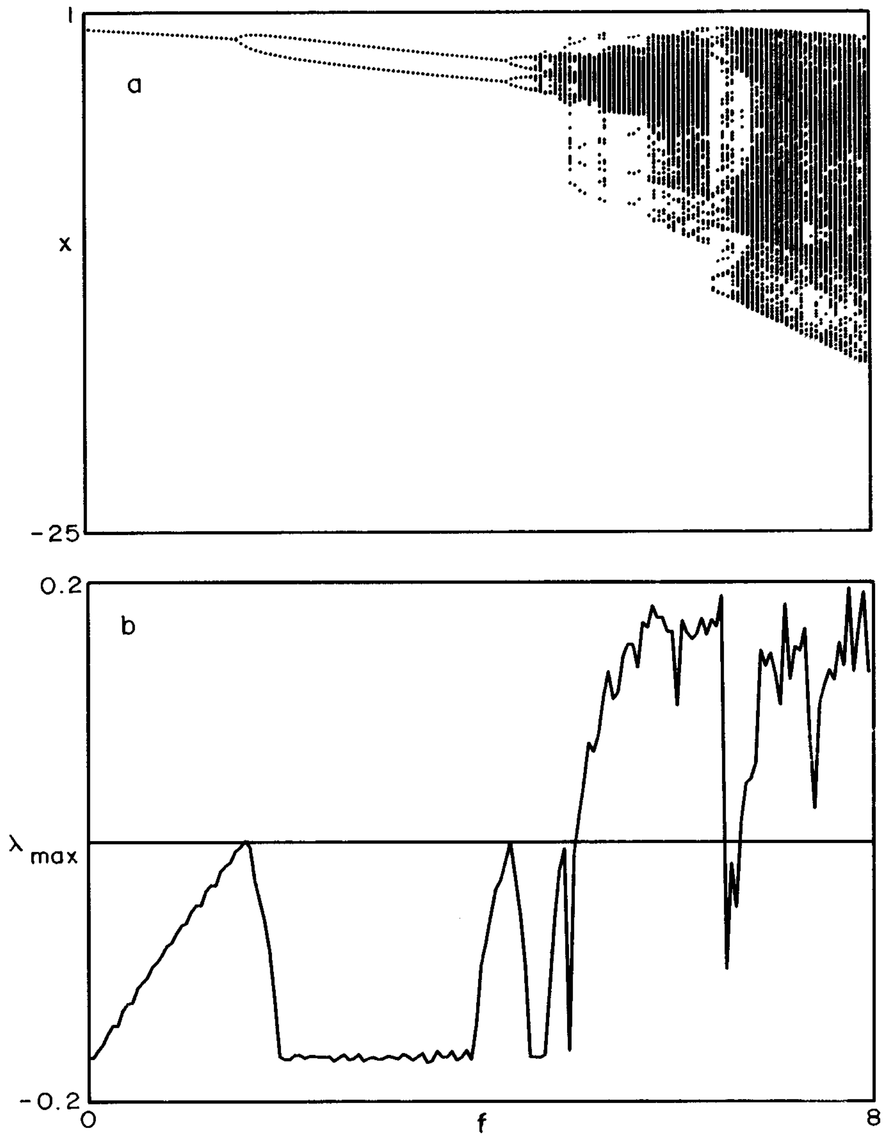

FIG. 2. Real-time dynamics of system (1) with the same parameter values as in Fig. 1. (a) Bifurcation diagram and (b) the maximal Lyapunov exponent (λ_{max}) as a function of the amplitude (f) of the external driving force.

From Eq. (33) we can study the singularity structure in the complex $z-$ plane by plotting Im(z) versus Re(z) for a chosen set of parametric values. As an illustration in Fig. 1(a), we have fixed the parameter values as $d=0.25$, $\alpha=1.0$, $\beta=1.0$, $\omega=2.0$ and obtained the singularity structure in the complex $z-$ plane about the singularity located at the origin (since we take $t_0=0$) for $f=0.5$. This singularity pattern, given by Eq. (33), can be mapped back to the complex $t-$ plane by the multivalued transformation [c.f. Eq. (23), where we have chosen $t_0=0$]

$$z=t^2\ln t \tag{34}$$

similar to the procedure adopted by Fournier, Levine, and Tabor [7] for the Duffing oscillator. This can be performed by using polar coordinates in both the $z-$ and $t-$ planes as

$$z=\rho e^{i\phi} \tag{35}$$

and

$$t=re^{i\theta}. \tag{36}$$

From (34) and (35) we can write the real and imaginary parts of z in terms of r and θ as

$$\mathrm{Re}\ (z)=r^2[\cos(2\theta)\ln r-(\theta+2\pi n)\sin(2\theta)], \tag{37a}$$

$$\mathrm{Im}\ (z)=r^2[\sin(2\theta)\ln r+(\theta+2\pi n)\cos(2\theta)], \tag{37b}$$

where n is the Riemann sheet index in the $t-$ plane. From this pair of equations (37), it follows that

$$r=\exp[-(\theta+2\pi n)\cot(2\theta-\phi)] \tag{38}$$

and so

$$\rho=-(\theta+2\pi n)\exp[-2(\theta+2\pi n)\cot(2\theta-\phi)] \times \mathrm{cosec}(2\theta-\phi). \tag{39}$$

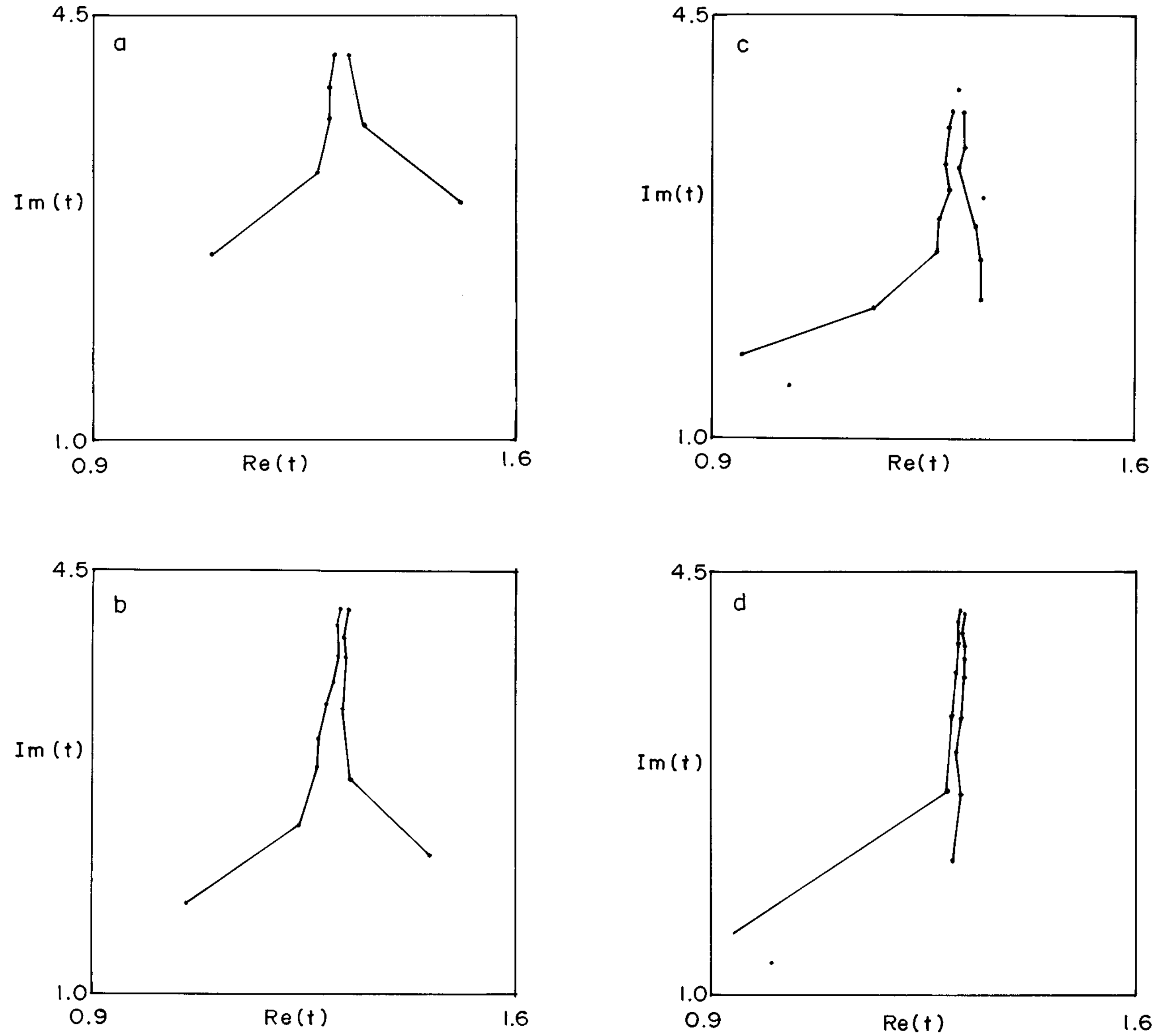

FIG. 3. Global singularity structure in the complex time plane of system (1) obtained numerically by using the ATOMFT package for (a) $f=0.5$, (b) $f=2.5$, (c) $f=4.5$, and (d) $f=6.5$, keeping the other parameter values the same as in Fig. 1.

Equations (38) and (39) completely determine the mapping $z\rightarrow$ t.

For a given pole in the $z-$ plane given by Eq. (33), we assign polar coordinates ρ and ϕ, which can be readily computed. Then from Eq. (39) we can compute the value of θ by a simple numerical root search method, for any sheet n, corresponding to the given (ρ,ϕ) values. From this value of θ the associated r value is computed from Eq. (38). Thus for any one of the singularities in the $z-$ plane given by Eq. (33), we can obtain the corresponding singularity and its substructure in the complex $t-$ plane through the analytic mapping (38) and (39). In Fig. 1(b) we have shown one such local singularity structure in the complex $t-$ plane, in the neighborhood of the marked singularity in Fig. 1(a), determined from the analytic mapping for the same choice of parametric values as mentioned above. From Fig. 1(b) we find that the local singularity structure obtained is a two-armed structure with the singularities becoming densely ''packed'' and clustered along each arm as they approach the center of the two arms, with n increasing. The recursive nature of this clustering leads to an immensely complicated singularity structure in the complex $t-$ plane.

Now we study numerically, the real-time chaotic dynamics exhibited by the damped driven Toda oscillator and then we analyze the corresponding singularity structure exhibited by the solution of the system (1) in the complex time plane. For our numerical study, we rewrite Eq. (1) as a system of three coupled first-order differential equations

$$\dot{x}=y,\quad \dot{y}=-dy-\alpha e^{x}+\beta+f\cos z,\quad \dot{z}=\omega. \qquad (40)$$

Then we integrate Eq. (40) numerically using the fourth-order Runge Kutta method for various values of f, keeping the other parameters fixed as $d=0.25$, $\alpha=1.0$, $\beta=1.0$, and

$\omega = 2.0$ and investigate the real-time dynamics of the system. The bifurcation diagram shown in Fig. 2(a) depicts the period-doubling route to chaos exhibited by the system. Figure 2(b) shows the corresponding largest Lyapunov exponent [16] $\lambda_{\max}$ as a function of increasing f, confirming the behavior of ordered and chaotic motions exhibited by the system as shown in Fig. 2(a).

Now let us investigate the corresponding global singularity structure pattern in the complex time plane, obtained numerically by using the ATOMFT package developed by Chang [17], as a function of the control parameter f. We concentrate on the singularity pattern formed by the singularities located by the ATOMFT program, only along the specified integration path, for various f values, while keeping the other parameters fixed (as in Fig. 2). The integration path is chosen such that in the complex time domain, we initially integrate along the path from the first leg (0.0,0.0) of the path up to the vertex (1.3,2.1) and then continue the path vertically up to the vertex (1.31,4.2). The same integration path is used in various f values used for our investigation.

For $f = 0.0$, the singularity structure of Eq. (1) in the complex time plane exhibits a simple deformed lattice pattern of singularities that corresponds to the damped oscillations in the real-time domain. When the value of f is increased to 0.5, the singularity structure is observed as shown in Fig. 3(a) and, correspondingly, in the real-time domain, the system exhibits a period-T limit-cycle behavior. The continuous lines that connect the singularity positions (depicted by the dots) in Fig. 3(a) are just to show clearly the observed chimney pattern of singularities in the complex $t-$ plane. Further increase to $f = 2.5$ and $f = 4.5$ results in the reduction of the width of the chimney pattern compared to that of Fig. 3(a), along with the fact that the singularities tend to accumulate at the top of the chimney pattern as shown, respectively, in Figs. 3(b) and 3(c). For $f = 6.5$, the system (1) exhibits chaotic oscillations (c.f. Fig. 2) in the real-time domain and the corresponding singularity structure in the complex time plane shows the clustering of singularities at the top of the chimney pattern and is shown in Fig. 3(d).

To conclude, the integrability and nonintegrability properties of the damped driven Toda oscillator, which has nonpolynomial terms in its equation of motion, has been investigated by analyzing the singularity structure in the complex time plane through Painlevé analysis. We identify the parameter choice $\beta = -2d^2$ when $f = 0$ for which the system (1) possesses the Painlevé property and we present the corresponding exact solutions. In the nonintegrable regime, we find that the local singularity structure of the system exhibits a two-armed infinite-sheeted Riemann structure of singularities in the complex $t-$ plane. Further, we have also analyzed numerically the global singularity structure of the system and compared it with their corresponding real-time chaotic dynamics exhibited by these systems. From our investigations, we observe that the global singularity pattern of the system in the complex time plane exhibits a chimneylike pattern whose width is reduced, and the singularities tend to accumulate at the top of the chimney pattern, as these systems undergo period doubling bifurcations leading to chaotic oscillations in the real-time domain. As the results reported here appear, to be common to those systems in which logarithmic singularities enter into their solution, we feel that more examples of this type may give us an indication of the generality of the above kind of results.

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